

SYMPLECTIC GENERIC COMPLEX STRUCTURES ON 4-MANIFOLDS WITH $b_+ = 1$

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ABSTRACT. We study symplectic structures on Kähler surfaces with $p_g = 0$. We give an example of a projective surface which admits a symplectic structure which is not compatible with any Kähler metric.

1. INTRODUCTION

The main purpose of this note is to give a negative answer to a question raised by Tian-Jun Li [Li08]:

Question 1.1. *Let X be a closed, smooth, oriented 4-manifold which underlies a Kähler surface such that $p_g(X) = 0$. Does X admit a symplectic generic complex structure?*

A complex structure J on X is called *symplectic generic* if for any symplectic form ω of X such that $-c_1(X, \omega)$ coincides with the canonical class K_J of J , there exists a Kähler form ω' cohomologous to ω .

One of the main motivations for this question is the fact that, by a result of Biran [Bir99], the existence of a symplectic generic complex structure on any rational 4-manifold implies the famous Nagata's conjecture (see [Li08] for more details). Recall that a smooth 4-manifold X is said to be *rational* if it is diffeomorphic to either $S^2 \times S^2$ or $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, for some $k \geq 0$.

On the other hand, if X is the 4-manifold underlying a smooth minimal projective surface of general type (i.e. with big and nef canonical line bundle) then there exists a symplectic form inside the class of the canonical line bundle of X (see [Cat09, STY02]). Therefore, if $p_g(X) = 0$, the existence of a symplectic generic complex structure on X would, in particular, imply the existence of a Kähler-Einstein metric with negative curvature on X , by the result of Aubin and Yau. For example, Catanese and LeBrun [CL97] showed the existence of a Kähler-Einstein metric with negative curvature on the generic Barlow surface, which is a projective surface of general type homeomorphic to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$. But the question remains a hard problem in general, as a classification of the projective surfaces with zero genus is still beyond our reach (see the recent survey [BCP10] for an updated account).

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Our example is obtained by considering the 4-manifold $X = (\Sigma \times S^2) \# \overline{\mathbb{CP}^2}$, where Σ is a Riemannian surface of genus one. We show the existence of a symplectic form on X which is not cohomologous to any Kähler form on X , with respect to any complex structure J . From an algebraic geometric point of view, this corresponds to saying that the Seshardi constant of a suitable ample class on any uniruled projective surface over an elliptic curve is not maximal (e.g. see [Gar06]). In particular, it follows that X does not admit a symplectic generic complex structure.

Moreover, we describe a minimal surface of general type, for which the underlying manifold does not admit a symplectic generic complex structure. The construction relies on a recent result by Bauer and Catanese [BC09].

Note that both these examples have infinite fundamental group.

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2. PRELIMINARY RESULTS

In this section, we recall some basic definition and well known facts about the space of symplectic forms on a smooth 4-manifold.

Given a closed smooth oriented 4-manifold X , we consider the *positive cone* of X , which is defined as the set

$$\mathcal{P}_X = \{a \in H^2(X, \mathbb{R}) \mid a^2 > 0\}.$$

Moreover, we denote by Ω_X the space of orientation-compatible symplectic forms on X . Let

$$\mathcal{C}_X = \{[\omega] \mid \omega \in \Omega_X\} \subseteq H^2(X, \mathbb{R})$$

and let $K_\omega = -c_1(X, \omega)$ be the canonical class of $\omega \in \Omega_X$. We denote by \mathcal{K}_X the union of all elements K_ω in $H^2(X, \mathbb{Z})$, where $\omega \in \Omega_X$. For any $K \in \mathcal{K}_X$, let

$$\mathcal{C}_{(X, K)} = \{[\omega] \in \mathcal{C}_X \mid K_\omega = K\}.$$

If K is a torsion class, then we replace $\mathcal{C}_{(X, K)}$ by its intersection with the component of \mathcal{P} which contains any Kähler class. Note that a complex structure J on X is symplectic generic if $\mathcal{C}_J = \mathcal{C}_{(X, K_J)}$, where \mathcal{C}_J denotes the Kähler cone of J and K_J is the canonical class of J .

Let \mathcal{E}_X be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection -1 . In particular, X is said to be *minimal* if \mathcal{E}_X is empty. Moreover, for any $K \in H^2(X, \mathbb{Z})$, we denote

$$\mathcal{E}_{(X, K)} = \{E \in \mathcal{E}_X \mid E \cdot K = -1\}.$$

The following result by Li and Luo [LL01] will play an important role:

Theorem 2.1. *Let (X, ω) be a closed, symplectic 4-manifold with $b_+(X) = 1$.*

Then

$$\mathcal{C}_X = \{a \in \mathcal{P}_X \mid a \cdot E \neq 0 \text{ for all } E \in \mathcal{E}_X\}.$$

Let $K \in \mathcal{K}_X$. Then $\mathcal{C}_{(X,K)}$ is contained in one of the components of \mathcal{P}_X , denoted by $\mathcal{P}_{(X,K)}$. Moreover,

$$\mathcal{C}_{(X,K)} = \{a \in \mathcal{P}_{(X,K)} \mid a \cdot E > 0 \text{ for all } E \in \mathcal{E}_{(X,K)}\}.$$

Proof. See [LL01, Theorem 4] and [Li08, Theorem 3.11]. \square

Lemma 2.2. *Let (X, J) be a minimal complex surface with $b_+(X) = 1$ and which admits a Kähler class $[\omega] \in \mathcal{C}_J$. Then J is a symplectic generic complex structure if and only if any J -holomorphic curve in X has non-negative self-intersection.*

Proof. By the Kähler Nakai-Moishezon criterion [Buc99, Lam99], if the Kähler cone \mathcal{C}_J is not empty then it coincides with the set of elements in $\mathcal{P}_{(X,K_J)}$ which are positive on every J -holomorphic curve with negative self-intersection. Thus, if there is no such a curve on X , it follows that J is a symplectic generic complex structure.

Let us assume now that C is a J -holomorphic curve with negative self-intersection. Let $v = \omega(C)$ and $m = -C^2$ and define $a(t) = [\omega] + tPD(C) \in H^2(X, \mathbb{R})$ for any $t \geq 0$. Then, since

$$a(t)^2 = [\omega]^2 + 2t\omega(C) + t^2C^2 > 2tv - t^2m,$$

it follows that there exists $T > v/m$ such that $a(T) \in \mathcal{P}_{(X,K_J)}$. Since X is minimal, Theorem 2.1 implies that $a(T)$ is represented by a symplectic form ω_T such that $K_{\omega_T} = K_J$. On the other hand, $\omega_T(C) = v - Tm < 0$, thus $a(T)$ is not a Kähler class. In particular, J is not a symplectic generic complex structure. \square

By the Kähler Nakai-Moishezon criterion and Theorem 2.1, it also follows that a positive answer to Question 1.1 in the case of rational 4-manifolds is equivalent to the following conjecture (Harbourne-Hirschowitz): any integral curve with negative self-intersection on the blow-up of $\mathbb{C}P^2$ at a set of points in very general position is a smooth rational curve with self-intersection -1 .

3. RULED MANIFOLDS

In this section, we show the existence of a smooth uniruled complex manifold, which does admit a symplectic generic complex structure.

Lemma 3.1. *Let Σ be an elliptic curve, and let $p: Y \rightarrow \Sigma$ be a minimal ruled surface over Σ , such that the parity of the intersection pairing on $H^2(Y, \mathbb{Z})$ is odd. Let X be the blow-up of Y at one point $\eta \in Y$. Let k be the canonical class of X , and let e be the class of the exceptional divisor.*

Then the class $e - 2k$ contains an effective curve.

Proof. By Atiyah's classification [Ati57] of rank 2 vector bundles on an elliptic curve, it follows that $Y = \mathbb{P}(\mathcal{E})$ where \mathcal{E} is either the indecomposable vector bundle contained in the sequence

$$0 \rightarrow \mathcal{O}_\Sigma \rightarrow \mathcal{E} \rightarrow \mathcal{O}_\Sigma(p) \rightarrow 0$$

for some $p \in \Sigma$ or $\mathcal{E} = \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(L)$ where L is a line bundle of odd degree $m < 0$.

Let us consider first the case of the indecomposable vector bundle. It is known (e.g. see [CC93]) that in this case $\mathbb{P}(\mathcal{E})$ is isomorphic to the symmetric product $S^2\Sigma$ of the elliptic curve Σ , i.e. the quotient of $\Sigma \times \Sigma$ by the natural action of $\mathbb{Z}/2\mathbb{Z}$. We will denote by $[x, y] \in S^2\Sigma$ the class of an element $(x, y) \in \Sigma \times \Sigma$. Note that the projection $p: S^2\Sigma \rightarrow \Sigma$ is defined by $p([x, y]) = x + y$. Consider the family of curves

$$C_t = \{[x, t+x] \mid x \in \Sigma\} \quad \text{for any } t \in \Sigma.$$

If $t \in \Sigma$ is not a 2-torsion point, then the curve C_t is a smooth elliptic curve. Otherwise, C_t is a non-reduced elliptic curve. Note that, for any $s, t \in \Sigma$, we have $C_t = C_s$ if and only if $t = s$ or $t = -s$ and C_t and C_s are disjoint otherwise. It follows that $C_t^2 = 0$. Moreover, given $s, t \in \Sigma$, there exist exactly 4 points $x \in \Sigma$ such that $2x + t = s$. Thus, if t is a general point in Σ , then the general fiber of p meets C_t in exactly 4 points. Let f be the numerical class of the pull-back of the general fiber of p in X and let δ be the numerical class of the pull-back of C_t . Then

$$\delta^2 = C_t^2 = 0 \quad \delta \cdot e = 0 \quad \text{and} \quad \delta \cdot f = 4.$$

By adjunction, we have that $k \cdot \delta = -\delta^2 = 0$. Similarly, we have $k \cdot e = -1$ and $k \cdot f = -2$. Moreover, since e, f and k are a basis of $H^2(X, \mathbb{Q})$, it follows easily that $\delta = 2e - 2k$. For any point $\eta \in S^2\Sigma$ there exists $t \in \Sigma$ such that $\eta \in C_t$. If X is the blow-up of Y at η and C'_t is the proper transform of C_t in X , then C'_t is in the class of $(2 - q)e - 2k$, where $q \geq 1$ is the multiplicity of C_t at η . In particular, the class $e - 2k$ contains an effective curve, as claimed.

Let us consider now the case of a decomposable vector bundle $\mathcal{E} = \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(L)$ where L is a line bundle on Σ of odd degree $m < 0$. Then, there exists an holomorphic section C in Y such that $C^2 = m$. If ξ is the numerical class of the pull-back of C in X , it follows easily that $2\xi = e + mf - k$, where f is the pull-back of the general fiber of p . In particular, $e - 2k = 4\xi + (-2mf - e)$ is the class of a (possibly not irreducible) effective curve in X . \square

Remark 3.2. Note that the uniruled surface which is the projectivization of the decomposable vector bundle can be obtained as a deformation of the projectivization of the indecomposable one. Thus, in the proof of the previous lemma, the second case would follow immediately from the first one.

Lemma 3.3. *A complex surface X homeomorphic to $(\Sigma \times S^2) \# \overline{\mathbb{CP}^2}$, is bi-holomorphic to a blow up at a single point of a minimal ruled surface Y over an elliptic curve, such that the intersection pairing on $H^2(Y, \mathbb{Z})$ is odd.*

Proof. Recall that from the Enriques-Kodaira classification of complex surfaces, it follows that each complex surfaces with odd b_+ is Kähler, and that any algebraic surface of non-negative Kodaira dimension and zero holomorphic Euler characteristics is bi-meromorphic to a torus or a bi-elliptic surface. Since $b_+(X) = 1$, it follows that X is Kähler and $p_g(X) = 0$. Thus X is algebraic. Since $\pi_1(X) = \mathbb{Z}^2$ and $\chi(\mathcal{O}_X) = 0$, we conclude that X has Kodaira dimension $-\infty$.

By the classification of algebraic surfaces, it follows that if Y is the minimal model of X , i.e. the surface obtained after blowing-down all the holomorphic (-1) spheres on X , then Y is a uniruled surface over a Riemannian surface Σ . Since $b_1(Y) = b_1(X) = 2$, it follows that the genus of Σ is one. Moreover, since $b_2(X) = 3$, it follows that X is the blow-up of a ruled surface over an elliptic curve at a single point $p \in Y$. In particular X has exactly two holomorphic rational curves E_1 and E_2 with self-intersection -1 : one is the exceptional divisor of the blow-up map and the other is the strict transform of the rational fiber passing through the blown-up point. Assume that the intersection form on $H^2(Y, \mathbb{Z})$ has even parity. Let C be a curve on Y which pass through p and which meets the fiber of the fibration $Y \rightarrow \Sigma$ transversally at p . Then the strict transform of C in X has odd self-intersection and it does not intersect E_2 . Thus, after contracting E_2 we obtain a surface Y' such that the intersection form on $H^2(Y', \mathbb{Z})$ has odd parity. After replacing Y by Y' , we may assume that $H^2(Y, \mathbb{Z})$ has odd parity. \square

Lemma 3.4. *Let $\pi: Y \rightarrow \Sigma$ be a ruled projective surface over an elliptic curve Σ , such that $H^2(Y, \mathbb{Z})$ has odd parity. Let X be the blow up of Y at a single point. Let k be the class of the canonical class of X and let e_1, e_2 be the classes of the two rational curves of self-intersection -1 on X .*

Then $\mathcal{E}_{(X,k)} = \{e_1, e_2\}$.

Proof. Let e be a class in $H_2(X, \mathbb{Z})$ which can be represented by a smoothly embedded sphere in X such that $e^2 = -1$. Then e belongs to the kernel of $\pi_*: H_2(X, \mathbb{Z}) \rightarrow H_2(\Sigma, \mathbb{Z})$. This kernel is spanned by e_1 and e_2 and we deduce $e = \pm(ne_1 + (n-1)e_2)$ for some integer n . At the same time $e_1 \cdot k = e_2 \cdot k = -1$, since e_1, e_2 are the classes of exceptional curves on X . Thus, if $e \in \mathcal{E}_{(X,k)}$, then $e \cdot k = -1$ which implies $e = e_1$ or $e = e_2$. \square

Theorem 3.5. *Let Σ be a Riemann surface of genus 1, let $\Sigma \times S^2$ be the trivial S^2 -bundle on Σ and let $X = (\Sigma \times S^2) \# \overline{\mathbb{CP}}^2$.*

Then, for any complex structure J on X , there exists a symplectic form ω on X such that ω is not Kähler with respect to J . Moreover, X does not admit any symplectic generic complex structure.

Proof. Let J be a complex structure on X , let k be the canonical class of (X, J) and let e be the class of the exceptional divisor E of the contraction $X \rightarrow Y$, whose existence is guaranteed by Lemma 3.3. Let a be the first Chern class of an ample

line bundle on X . By Lemma 3.1, it follows that $v = a \cdot (e - 2k) > 0$. Let

$$a(t) = a + t(e - 2k) \in H^2(X, \mathbb{R}) \quad \text{for all } t > 0.$$

In particular, $a(t) \cdot (e - 2k) = v - t$ and $a(v)^2 = a^2 + v^2 > 0$. Thus, there exists $T > v$ such that $a(T)^2 > 0$. Moreover, if $E \in \mathcal{E}_{(X,k)}$, then

$$a \cdot E > 0 \quad k \cdot E = -1 \quad \text{and by Lemma 3.4} \quad e \cdot E \geq -1.$$

Thus, $a(t) \cdot E > 0$ for all $t > 0$. Since $b_+(X) = 1$, Theorem 2.1 implies that the class $a(T)$ is represented by a symplectic form ω , such that $K_\omega = k$.

On the other hand, by Lemma 3.1, the class $e - 2k$ is represented by a J -holomorphic curve C such that $a(T) \cdot C < 0$, since $T > v$. Thus, the class $a(T)$ does not contain a Kähler form. In particular, J is not a symplectic generic complex structure. \square

4. NON-RULED MANIFOLDS

In this section we study Question 1.1 in the case of smooth *minimal* 4-manifolds with non-negative Kodaira dimension.

Question 4.1. *Let X be a minimal 4-manifold which underlies a Kähler surface such that $p_g(X) = 0$. Does X admit a symplectic generic complex structure?*

In particular, we show that the question has positive answer in the case of zero Kodaira dimension and we provide an example of a minimal surface of general type which does not admit a symplectic generic complex structure.

By the Sieberg-Witten theory, the Kodaira dimension of a Kähler surface is preserved under diffeomorphism [BHPdV04]. As noted in [Li08], any uniruled 4-manifold, i.e. a manifold which underlies a Kähler surfaces of Kodaira dimension $-\infty$, admits a symplectic generic complex structure.

We first consider the case of zero Kodaira dimension:

Proposition 4.2. *Let X be a 4-manifold which underlies a Kähler surface such that $p_g(X) = 0$ and $\text{kod}(X) = 0$.*

Then X admits a symplectic generic complex structure.

Proof. By the classification of algebraic surfaces, it follows that the canonical class of X is numerically trivial. Thus, by the adjunction formula, the only holomorphic curves of negative self-intersection, are smooth rational curves C such that $C^2 = -2$. In particular, Lemma 2.2 implies that it is sufficient to show that there exists a complex structure on X which does not admit any of these curves.

By the classification of algebraic surfaces, we just need to consider two cases: Enriques surfaces and bi-elliptic surfaces. The moduli space of Enriques surfaces is irreducible and by a result of Barth and Peters [BP83, Proposition 2.8], the generic Enriques surface does not contain any smooth rational curve of self-intersection -2 .

If X is a bi-elliptic surface, then $X = \Sigma_1 \times \Sigma_2 / G$, where Σ_1 and Σ_2 are Riemannian surfaces of genus one and G is an abelian group acting by complex multiplication on Σ_1 and by translation on Σ_2 . Then the natural action of Σ_2 on $\Sigma_1 \times \Sigma_2$ commutes with the action of G and in particular Σ_2 acts on X non trivially. Thus, X does not admit any negative self-intersection curve. By Lemma 2.2, it follows that any complex structure on X is symplectic generic. \square

If X is a minimal surfaces of general type with $p_g(X) = 0$, it is well known that $q = 0$ and $1 \leq K_X^2 \leq 9$. Thus, their moduli spaces is a union of finitely many irreducible varieties. Nevertheless, it is still not clear what the topology for these surfaces is (see [BCP10] for a recent survey). As stated in the introduction, if X is the 4-manifold underlying the surface X , a positive answer to question 4.1 would imply the existence of a complex structure on X which admits a Kähler-Einstein metric. By the results in [Bar84, LP07, PPS09a, PPS09b], it follows that there exist a surface of general type which is homeomorphic to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, for $5 \leq k \leq 8$. It follows by [CL97, RŞ09] that, on any of these surfaces, there exists a complex structure which admits a Kähler-Einstein metric with negative curvature.

In general, if X is a minimal surface of general type with $p_g(X) = 0$, then $\chi(\mathcal{O}_X) = 1$ and by Noether's formula we have

$$b_2(X) = \chi(X) - 2 = 12\chi(\mathcal{O}_X) - K_X^2 - 2 = 10 - K_X^2.$$

Thus, if $K_X^2 = 9$, then any class in \mathcal{P}_X is the multiple of an ample class and the answer to Quesiton 4.1 is obvious.

Let us consider now the case of a surface of general type S with $p_g(X) = 0$ and $K_X^2 = 8$. All the known examples have infinite fundamental group and their universal cover is the bidisk $\Delta_1 \times \Delta_2 \subseteq \mathbb{C}^2$ [BCP10], so we assume that S is of this type. Denote by w_1 and w_2 two semi-positive (1,1)-forms on $\Delta_1 \times \Delta_2$ obtained via pullbacks of Poincaré metrics from the projections of the bidisk to its factors. For any $a, b > 0$ the form $aw_1 + bw_2$ is Kähler on the bidisk and is invariant under the action of $\pi_1(X)$. Thus, it descends to a Kähler form $w_{a,b}$ on S . Since $b_2(X) = 2$, it follows that for $a, b > 0$ the forms $w_{a,b}$ span one of the two connected components of \mathcal{P}_X , and so the complex structure on X is symplectic generic.

On the other hand, the results in [BC09] immediately imply the existence of a minimal surface of general type which does not admit a symplectic generic complex structure. Burniat showed the existence of a minimal surface X of general type such that $K_X^2 = 6$, $p_g(X) = 0$, and which admits a $(\mathbb{Z}/2\mathbb{Z})^2$ -cover of \mathbb{CP}^2 blown-up at 3 points. We will call such a surface a *Burniat surface*.

Theorem 4.3. *Let X be a 4-manifold which underlies a Burniat surface. Then X does not admit a symplectic generic complex structure.*

Proof. By [BC09, Theorem 0.2], any complex structure J on X is a Burniat surface. In particular, X admits a J -holomorphic curve C of negative self-intersection, which maps to a (-1) -curve on the blow-up of \mathbb{CP}^2 at 3 points. More specifically, C is an

elliptic curve of self-intersection -1 . Thus, by Lemma 2.2, it follows that J is not symplectic generic. \square

Note that a Burniat surface has infinite fundamental group. We do not know any complex surface with $p_g = 0$, finite fundamental group and which does not admit a symplectic generic complex structure.

Recall finally that there exist a wide class of minimal elliptic surfaces of Kodaira dimension 1 and with $p_g = 0$. These surfaces have topological Euler characteristic equal to 12, the base of the corresponding elliptic fibration is \mathbb{CP}^1 , and the fibration can have any number of multiple fibers greater than 1. It would be interesting to show that all such surfaces admit a symplectic generic complex structure.

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